

# STRUCTURE OF THE FUNDAMENTAL GROUPS OF ORBITS OF SMOOTH FUNCTIONS ON SURFACES

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ABSTRACT. Let  $M$  be a smooth compact connected surface,  $P$  be either the real line  $\mathbb{R}$  or the circle  $S^1$  and  $f : M \rightarrow P$  be a Morse map. Denote by  $\mathcal{S}(f)$  and  $\mathcal{O}(f)$  the corresponding stabilizer and orbit of  $f$  with respect to the right action of the group  $\mathcal{D}(M)$  of diffeomorphisms of  $M$ . In a series of papers the author described homotopy types of  $\mathcal{S}(f)$  and computed higher homotopy groups of  $\mathcal{O}(f)$ . The present paper describes the structure of the remained fundamental group  $\pi_1\mathcal{O}(f)$  for the case when  $M$  is orientable and differs from 2-sphere and 2-torus.

The result holds as well for a larger class of smooth maps  $f : M \rightarrow P$  having the following property: the germ of  $f$  at each of its critical points is smoothly equivalent to a homogeneous polynomial  $\mathbb{R}^2 \rightarrow \mathbb{R}$  without multiple factors.

## 1. INTRODUCTION

Let  $M$  be a smooth compact connected surface and  $P$  be either the real line  $\mathbb{R}$  or the circle  $S^1$ . For each closed subset  $X \subset M$  let  $\mathcal{D}(M, X)$  be the group of  $C^\infty$ -diffeomorphisms fixed on  $X$  and

$$\mathcal{S}(f, X) = \{h \in \mathcal{D}(M, X) \mid f \circ h = f\}, \quad \mathcal{O}(f, X) = \{f \circ h \mid h \in \mathcal{D}(M, X)\}$$

be respectively the *stabilizer* and the *orbit* of  $f \in C^\infty(M, P)$  under the standard right action of  $\mathcal{D}(M, X)$  on  $C^\infty(M, P)$ .

We will endow  $\mathcal{D}(M, X)$  and  $C^\infty(M, P)$  with  $C^\infty$  Whitney topologies. These topologies induce certain topologies on the spaces  $\mathcal{S}(f, X)$  and  $\mathcal{O}(f, X)$ . Denote by  $\mathcal{D}_{\text{id}}(M, X)$  and  $\mathcal{S}_{\text{id}}(f, X)$  the identity path components of  $\mathcal{D}(M, X)$  and  $\mathcal{S}(f, X)$  respectively and by  $\mathcal{O}_f(f)$  the path component of  $\mathcal{O}(f)$  containing  $f$ . If  $X = \emptyset$ , we will omit  $X$  from notation, e.g. we write  $\mathcal{D}(M)$  instead of  $\mathcal{D}(M, \emptyset)$ , and so on.

In [7, 8, 10, 11, 12] for a large class of smooth maps  $f : M \rightarrow P$  and certain “ $f$ -adopted submanifolds”  $X \subset M$  the author described the homotopy types of  $\mathcal{S}(f, X)$ , computed the higher homotopy groups of  $\mathcal{O}(f, X)$ , and obtained certain information about  $\pi_1\mathcal{O}(f, X)$ , see Theorem 1.4 below.

The main result of this paper, Theorem 1.10, gives a complete description of the structure of  $\pi_1\mathcal{O}(f, X)$  for the case when  $M$  is orientable and differs from 2-sphere and 2-torus. It expresses  $\pi_1\mathcal{O}(f, X)$  in terms of special wreath product with  $\mathbb{Z}$  over some finite cyclic groups.

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**1.1. Preliminaries.** Let  $\mathcal{C}_\partial^\infty(M, P)$  be the subset of  $\mathcal{C}^\infty(M, P)$  consisting of maps  $f$  satisfying the following axiom:

**Axiom (B).** *The map  $f : M \rightarrow P$  takes constant values on connected components of  $\partial M$ , and the set  $\Sigma_f$  of critical points of  $f$  is contained in the interior  $\text{Int}M$ .*

Denote by  $\text{Morse}(M, P) \subset \mathcal{C}_\partial^\infty(M, P)$  the subset consisting of *Morse* maps, i.e. maps having only non-degenerate critical points. It is well known that  $\text{Morse}(M, P)$  is open and everywhere dense in  $\mathcal{C}_\partial^\infty(M, P)$ .

Let also  $\mathcal{F}(M, P)$  be the subset of  $\mathcal{C}_\partial^\infty(M, P)$  consisting of maps  $f$  satisfying the following additional axiom:

**Axiom (L).** *For each critical point  $z$  of  $f$  the germ of  $f$  at  $z$  is smoothly equivalent to some homogeneous polynomial  $f_z : \mathbb{R}^2 \rightarrow \mathbb{R}$  without multiple factors.* By Morse lemma a non-degenerate critical point of a map  $f : M \rightarrow P$  is smoothly equivalent to a homogeneous polynomial  $\pm x^2 \pm y^2$  which, evidently, has no multiple factors, and so satisfies (L). This implies that

$$\text{Morse}(M, P) \subset \mathcal{F}(M, P).$$

Notice that every critical point satisfying Axiom (L) is isolated. Moreover such a point  $z$  is non-degenerate if and only if the corresponding homogeneous polynomial  $f_z$  has degree  $\geq 3$ , see e.g. [8, §7].

**Definition 1.2.** [12]. *Let  $f \in \mathcal{F}(M, P)$ ,  $X \subset M$  be a compact submanifold, not necessarily connected, and whose connected components may have distinct dimensions. Let also  $X^i$ ,  $i = 0, 1, 2$ , be the union of connected components of  $X$  of dimension  $i$ . Then  $X$  will be called ***f*-adopted** if the following conditions hold true:*

- (a)  $X \cap \Sigma_f \subset X^0 \cup \text{Int}X^2$ ;
- (b)  $f$  takes constant value at each connected component of  $X^1 \cup \partial X^2$ .

The following lemma gives examples of  $f$ -adopted submanifolds. We left it to the reader.

**Lemma 1.3.** *Let  $X, Y \subset M$  be two submanifolds.*

- 1) *If  $X$  is  $f$ -adopted, then so is every connected component of  $X$ .*
- 2) *If  $X$  and  $Y$  are  $f$ -adopted and disjoint, then  $X \cup Y$  is  $f$ -adopted as well.*
- 3) *Suppose every connected component of  $X$  has dimension 2. Then  $X$  is  $f$ -adopted if and only if the restriction  $f|_X$  satisfies axioms (B) and (L).*
- 4) *Let  $a, b \in P$  be two distinct regular values of  $f \in \mathcal{F}(M, P)$ , and  $[a, b] \subset P$  the closed segment between them. Then  $X = f^{-1}[a, b]$  and any family of connected components of  $X$  is  $f$ -adopted.*  $\square$

Let  $f \in \mathcal{F}(M, P)$  and  $X \subset M$  be an  $f$ -adopted submanifold. Denote

$$\mathcal{S}'(f) = \mathcal{S}(f) \cap \mathcal{D}_{\text{id}}(M), \quad \mathcal{S}'(f, X) = \mathcal{S}(f) \cap \mathcal{D}_{\text{id}}(M, X). \quad (1.1)$$

In a sequel all the homotopy groups of  $\mathcal{O}(f, X)$  will have  $f$  as a base point, and so the notation  $\pi_k \mathcal{O}(f, X)$  will always mean  $\pi_k(\mathcal{O}(f, X), f)$ . Notice that the latter group is also isomorphic with  $\pi_k(\mathcal{O}_f(f, X), f)$ . The following theorem summarizes the results concerning the homotopy types of  $\mathcal{S}_{\text{id}}(f, X)$  and  $\mathcal{O}_f(f, X)$ .

**Theorem 1.4.** [7, 10, 11, 12]. *Let  $f \in \mathcal{F}(M, P)$  and  $X \subset M$  be an  $f$ -adopted submanifold. Then the following statements hold true.*

- 1)  $\mathcal{O}_f(f, X) = \mathcal{O}_f(f, X \cup \partial M)$ .
- 2) The map  $p : \mathcal{D}(M, X) \longrightarrow \mathcal{O}(f, X)$  defined by  $p(h) = f \circ h$  is a Serre fibration.
- 3) Suppose that either  $f$  has at least one critical point being **not a non-degenerate local extreme** or  $M$  is non-orientable. Then  $\mathcal{S}_{\text{id}}(f)$  is contractible,  $\pi_n \mathcal{O}(f) = \pi_n M$  for  $n \geq 3$ ,  $\pi_2 \mathcal{O}_f(f) = 0$ , and we also have the following short exact sequence:

$$1 \longrightarrow \pi_1 \mathcal{D}_{\text{id}}(M) \xrightarrow{p} \pi_1 \mathcal{O}(f) \xrightarrow{\partial} \pi_0 \mathcal{S}'(f) \longrightarrow 1. \quad (1.2)$$

If  $M$  is orientable and distinct from  $S^2$  and  $T^2$  then  $\pi_1 \mathcal{O}(f)$  is solvable.

- 4) Suppose that the Euler characteristic  $\chi(M)$  is less than the number of points in  $X$ . This holds for instance when either  $\dim X > 0$  or  $\chi(M) < 0$ . Then both  $\mathcal{D}_{\text{id}}(M, X)$  and  $\mathcal{S}_{\text{id}}(f, X)$  are contractible,  $\pi_i \mathcal{O}(f, X) = 0$  for  $i \geq 2$ , and we have an isomorphism:

$$\pi_1 \mathcal{O}(f, X) \cong \pi_0 \mathcal{S}'(f, X).$$

Moreover, there exist finitely many mutually disjoint  $f$ -adopted subsurfaces  $B_1, \dots, B_n \subset M \setminus (X^1 \cup X^2)$  each diffeomorphic either to a 2-disk  $D^2$ , or a cylinder  $S^1 \times I$ , or a Möbius band  $Mo$ , and such that if we denote

$$f_i := f|_{B_i} : B_i \rightarrow P, \quad \hat{B}_i := (B_i \cap X^0) \cup \partial B_i$$

for  $i = 1, \dots, n$ , then the following isomorphisms hold:

$$\pi_1 \mathcal{O}_f(f, X) \cong \pi_0 \mathcal{S}'(f, X) \cong \bigtimes_{i=1}^n \pi_0 \mathcal{S}'(f_i, \hat{B}_i) \cong \bigtimes_{i=1}^n \pi_0 \mathcal{O}_{f_i}(f_i, \hat{B}_i).$$

- 5) Let  $U$  be any open neighbourhood of  $X^1 \cup X^2$ . Then there exists an  $f$ -adopted submanifold  $N \subset M$  such that every connected component of  $N$  has dimension 2,

$$X^0 \cap N = \emptyset, \quad X^1 \cup X^2 \subset \text{Int} N \subset N \subset U$$

and the inclusion  $\mathcal{S}'(f, X^0 \cup N) \subset \mathcal{S}'(f, X)$  is a homotopy equivalence.

*Proof.* Statements 1) and 5) are proved in [12, Corollaries 2.1 & 6.2] respectively.

Statement 2) is a general result initially established in the paper by F. Sergeraert [15] for smooth functions of *finite codimension* on arbitrary closed manifolds. In particular, all singularities satisfying Axiom (L) have finite codimensions, [11, Lemma 12]. This covers the case  $X = \emptyset$ . The proof for  $X = \Sigma_f$  was given in [7, §11], and for arbitrary  $f$ -adopted submanifold  $X$  in [12, Theorem 5.1].

Statement 3) is proved in [7, Theorems 1.3, 1.5] for Morse maps, and extended to the class  $\mathcal{F}(M, P)$  in [11]. Solvability result is obtained in [13].

Statement 4) was initially established in [10, Theorem 1.7] for  $X = \emptyset$ , and extended to the general case in [12, Theorem 2.4].  $\square$

**1.5. Wreath products  $A \wr_{\mathbb{Z}_m} \mathbb{Z}$  and  $A \wr_{\mathbb{Z}_m} \mathbb{Z}_m$ .** Let  $A$  be any group and  $\mathbb{Z}_m$ ,  $m \geq 1$ , be a finite cyclic group of order  $m$ . Denote by  $A^{\mathbb{Z}_m}$  the set of *all maps*  $\mathbb{Z}_m \rightarrow A$  (being not necessarily homomorphisms). Then  $\mathbb{Z}_m$  naturally acts on  $A^{\mathbb{Z}_m}$  from the right and therefore one can define the corresponding semidirect product  $A^{\mathbb{Z}_m} \rtimes \mathbb{Z}_m$  which is denoted by

$$A \wr \mathbb{Z}_m$$

and called *wreath product* of  $A$  and  $\mathbb{Z}_m$ .

More generally, notice that the group  $\mathbb{Z}$  acts on  $\mathbb{Z}_m$  by the rule

$$z * k = z + k \pmod{m}$$

for  $z \in \mathbb{Z}$  and  $k \in \mathbb{Z}_m$ . This action induces a right action of  $\mathbb{Z}$  on  $A^{\mathbb{Z}_m}$  and therefore one can define the corresponding semidirect product  $A^{\mathbb{Z}_m} \rtimes \mathbb{Z}$  which will be denoted by

$$A \wr_{\mathbb{Z}_m} \mathbb{Z}$$

and called the *wreath product* of  $A$  and  $\mathbb{Z}$  over  $\mathbb{Z}_m$ .

Thus  $A \wr_{\mathbb{Z}_m} \mathbb{Z}$  is the set  $A^{\mathbb{Z}_m} \times \mathbb{Z}$  with the multiplication defined as follows. Let  $(\alpha, a), (\beta, b) \in A^{\mathbb{Z}_m} \times \mathbb{Z}$ . Define  $\gamma : \mathbb{Z}_m \rightarrow A$  by the formula  $\gamma(i) = \alpha(i + b) \cdot \beta(i)$ ,  $i \in \mathbb{Z}_m$ , where  $\cdot$  is the multiplication in  $A$ . Then, by definition, the product  $(\alpha, a)$  and  $(\beta, b)$  in  $A \wr_{\mathbb{Z}_m} \mathbb{Z}$  is

$$(\alpha, a)(\beta, b) := (\gamma, a + b).$$

If  $\epsilon : \mathbb{Z}_m \rightarrow A$  is the constant map into the unit of  $A$ , then  $(\epsilon, 0)$  is the unit of  $A \wr_{\mathbb{Z}_m} \mathbb{Z}$ .

Evidently, for  $m = 1$  the group  $A \wr_{\mathbb{Z}_m} \mathbb{Z}$  is isomorphic with the direct product  $A \times \mathbb{Z}$ . Also if  $A = \{1\}$ , then  $A \wr_{\mathbb{Z}_m} \mathbb{Z} \cong \mathbb{Z}$  for all  $m \geq 1$ . Finally notice that there is a natural *epimorphism*

$$q : A \wr_{\mathbb{Z}_m} \mathbb{Z} \longrightarrow A \wr_{\mathbb{Z}_m} \mathbb{Z}_m, \quad q(\alpha, n) = (\alpha, n \bmod m).$$

**1.6. Action of  $\mathcal{S}(f)$  on the Kronrod-Reeb graph.** For  $f \in \mathcal{F}(M, P)$  denote by  $\Gamma(f)$  the *Kronrod-Reeb graph* of  $f$ , i.e. the factor-space of  $M$  obtained by shrinking every connected component of every level-set  $f^{-1}(c)$  of  $f$  to a point. This graph is very useful for understanding the structure of  $f$ , see e.g. [4, 1, 3, 16].

Notice that there is a natural action of  $\mathcal{S}(f)$  on  $\Gamma(f)$  defined as follows. Let  $h \in \mathcal{S}(f)$ , so  $f \circ h = f$ . Then  $h(f^{-1}(c)) = f^{-1}(c)$  for all  $c \in P$ . In particular,  $h$  interchanges connected components of  $f^{-1}(c)$  being points of  $\Gamma(f)$ , and therefore it yields a certain homeomorphism  $\lambda(h)$  of  $\Gamma(f)$ , such that the correspondence  $h \mapsto \lambda(h)$  is a homomorphism  $\lambda : \mathcal{S}(f) \rightarrow \text{Aut}(\Gamma(f))$  into the group of all automorphisms of  $\Gamma(f)$ . Let

$$\mathbf{G}(f) := \lambda(\mathcal{S}'(f))$$

be the group of automorphisms of  $\Gamma(f)$  induced by isotopic to the identity diffeomorphisms of  $M$  preserving  $f$ .

**Definition 1.7.** [13]. *Let  $\mathcal{R}$  be the minimal class of all finite groups satisfying the following conditions:*

- (1) *the unit group  $\{1\}$  belongs to  $\mathcal{R}$ ;*
- (2) *if  $A, B \in \mathcal{R}$  then  $A \times B \in \mathcal{R}$ ;*
- (3) *if  $A \in \mathcal{R}$  and  $m \geq 1$ , then  $A \wr_{\mathbb{Z}_m} \mathbb{Z} \in \mathcal{R}$ .*

**Theorem 1.8.** [13]. *Let  $M$  be a compact orientable surface distinct from  $S^2$  and  $T^2$ . Then the class  $\mathcal{R}$  coincides with each of the following classes of groups:*

$$\{ \mathbf{G}(f) \mid f \in \text{Morse}(M, P) \}, \quad \{ \mathbf{G}(f) \mid f \in \mathcal{F}(M, P) \}.$$

In other words, for each  $f \in \mathcal{F}(M, P)$  the group  $\mathbf{G}(f)$  can be obtained from the unit group  $\{1\}$  by finitely many operations of direct products and wreath products from the top with certain finite cyclic groups. Conversely, for any group  $G \in \mathcal{R}$  one can find  $f \in \mathcal{F}(M, P)$ , which can be assumed even Morse, such that  $G \cong \mathbf{G}(f)$ .

Since  $\mathbf{G}(f)$  is a finite group,  $\lambda$  reduces an *epimorphism*  $\lambda : \pi_0 \mathcal{S}'(f) \rightarrow \mathbf{G}(f)$ . Also notice that we have a *surjective* boundary homomorphism  $\partial_1 : \pi_1 \mathcal{O}_f(f) \rightarrow \pi_0 \mathcal{S}'(f)$ , see Eq. (1.2). Therefore

$$\mathbf{G}(f) = \lambda \circ \partial_1(\pi_1 \mathcal{O}(f))$$

is a factor group of  $\pi_1 \mathcal{O}(f)$ . Thus Theorem 1.8 says that the factor group  $\mathbf{G}(f)$  of  $\pi_1 \mathcal{O}(f)$  can be described in terms of wreath products  $A \wr_{\mathbb{Z}_m}$  being factor groups of  $A \wr \mathbb{Z}$ .

Our main result, Theorem 1.10 below, shows that  $\pi_1 \mathcal{O}(f)$  itself can be described in terms of wreath products  $A \wr_{\mathbb{Z}_m}$ .

**Definition 1.9.** *Let  $\mathcal{P}$  be the minimal class of groups satisfying the following conditions:*

- (1) *the unit group  $\{1\}$  belongs to  $\mathcal{P}$ ;*
- (2) *if  $A, B \in \mathcal{P}$ , then  $A \times B \in \mathcal{P}$ ;*
- (3) *if  $A \in \mathcal{P}$  and  $m \geq 1$ , then  $A \wr_{\mathbb{Z}_m} \in \mathcal{P}$ .*

**Theorem 1.10.** *Let  $M$  be a connected compact orientable surface distinct from 2-sphere and 2-torus. Then the class  $\mathcal{P}$  coincides with each of the following two classes of fundamental groups:*

$$\{ \pi_1 \mathcal{O}(f) \mid f \in \text{Morse}(M, P) \}, \quad \{ \pi_1 \mathcal{O}(f) \mid f \in \mathcal{F}(M, P) \}.$$

It means that for each  $f \in \mathcal{F}(M, P)$  the group  $\pi_1 \mathcal{O}(f)$  can be obtained from  $\{1\}$  by finitely many operations of direct products and wreath products from the top with  $\mathbb{Z}$  over certain finite cyclic groups. Conversely, for any group  $G \in \mathcal{P}$  one can find  $f \in \mathcal{F}(M, P)$ , which can be assumed even Morse, such that  $G \cong \pi_1 \mathcal{O}(f)$ .

The proof will be given in §4.

**Remark 1.11.** If  $f$  is generic, that is every critical level set of  $f$  contains exactly one critical point, then by [7],  $\mathbf{G}(f) = \{1\}$  and  $\pi_1 \mathcal{O}(f) \cong \mathbb{Z}^k$  for some  $k \geq 0$ . In particular,  $\pi_1 \mathcal{O}(f) \in \mathcal{P}$ .

**Remark 1.12.** It is proved in [13] that each group  $A \in \mathcal{R}$  is solvable. By similar arguments the same statement can be established for the class  $\mathcal{P}$ . Therefore Theorem 1.10 gives another proof of solvability result in 3) of Theorem 1.4. We leave the details for the reader.

**Remark 1.13.** Theorem 1.10 holds for certain classes of smooth functions on  $T^2$ , see [14].

## 2. CRITICAL LEVEL SETS

Let  $f \in \mathcal{F}(M, P)$ ,  $c \in P$  and  $K$  be a connected component of the level set  $f^{-1}(c)$ . We will call  $K$  *critical* if it contains critical points of  $f$ , and *regular* otherwise.

Assume that  $K$  is critical. Then due to Axiom (L)  $K$  has a structure of a 1-dimensional CW-complex whose 0-cells are critical points of  $f$  belonging to  $K$ .

Choose  $\varepsilon > 0$  and let  $N$  be a connected component of  $f^{-1}[c - \varepsilon, c + \varepsilon]$  containing  $K$ . We will call  $N$  an  *$f$ -regular neighbourhood* of  $K$  if the following two conditions hold, see Figure 2.1:

$$N \cap \partial M = \emptyset, \quad N \cap \Sigma_f = K \cap \Sigma_f.$$

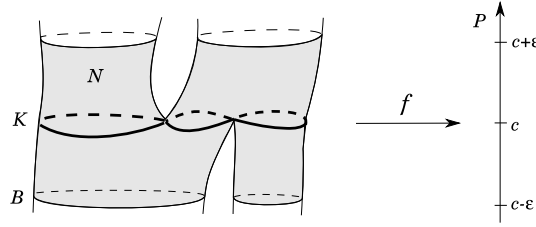


FIGURE 2.1. Critical component of level-set of  $f$

Let also

$$\mathcal{S}_{\text{inv}}(f, K) = \{h \in \mathcal{S}(f) \mid h(K) = K\}$$

be the subgroup of  $\mathcal{S}(f)$  consisting of diffeomorphism leaving  $K$  invariant. We will now define two equivalence relations  $\sim_K$  and  $\sim_{\partial N}$  on  $\mathcal{S}_{\text{inv}}(f, K)$ .

Given a homeomorphism  $h : M \rightarrow M$  and a *connected orientable submanifold*  $X \subset M$  we will say that  $X$  is *positively invariant* for  $h$  whenever the following conditions hold true:

- $h(X) = X$ , and
- if  $X$  is not a point, then the restriction map  $h|_X : X \rightarrow X$  is a preserving orientation homeomorphism.

Let  $h \in \mathcal{S}_{\text{inv}}(f, K)$ . Since  $h$  preserves the set of critical points of  $f$ , it follows that the restriction  $h|_K : K \rightarrow K$  is a cellular homeomorphism of  $K$ . We will say that  $h$  is  *$K$ -trivial*, if each cell of  $K$  is positively invariant for  $h$ . Denote by  $\mathcal{T}(f, K)$  the subgroup of  $\mathcal{S}_{\text{inv}}(f, K)$  consisting of  $K$ -trivial diffeomorphisms. Evidently,  $\mathcal{T}(f, K)$  is normal in  $\mathcal{S}_{\text{inv}}(f, K)$ . Given  $g \in \mathcal{S}_{\text{inv}}(f, K)$  we will write  $g \sim_K h$  whenever  $g^{-1} \circ h \in \mathcal{T}(f, K)$ .

Furthermore, let  $N$  be an  $f$ -regular neighbourhood of  $K$ . Then  $h(N) = N$  and so  $h$  yields a certain permutation of connected components of  $\partial N$ . Say that  $h$  is  *$\partial N$ -trivial*, if each connected component of  $\partial N$  is positively invariant for  $h$ . Denote by  $\mathcal{T}(f, \partial N)$  the normal subgroup of  $\mathcal{S}_{\text{inv}}(f, K)$  consisting of  $\partial N$ -trivial diffeomorphisms. Again for  $g \in \mathcal{S}_{\text{inv}}(f, K)$  we write  $g \sim_{\partial N} h$  whenever  $g^{-1} \circ h \in \mathcal{T}(f, \partial N)$ .

**Lemma 2.1.** see [7, Theorems 6.2, 7.1]. *Let  $g, h \in \mathcal{S}_{\text{inv}}(f, K)$  and  $N$  be an  $f$ -regular neighbourhood of  $K$ . Then the following statements hold.*

- (1) Suppose there exists at least one edge  $\delta$  being positively invariant for  $g^{-1} \circ h$ . Then all cells of  $K$  are also positively invariant for  $g^{-1} \circ h$ , and in particular,  $g \underset{K}{\sim} h$ .
- (2) Let  $W$  be an open neighbourhood of  $N$ . If  $g \underset{K}{\sim} h$ , then there exists an isotopy of  $g$  in  $\mathcal{S}_{\text{inv}}(f, K)$  supported in  $W$  to some  $g'$  such that  $g' = h$  on  $N$ .  
In particular, every  $K$ -trivial diffeomorphism is isotopic in  $\mathcal{S}_{\text{inv}}(f, K)$  via an isotopy supported in  $W$  to a diffeomorphism fixed on  $N$ .
- (3)  $\mathcal{T}(f, K) \subset \mathcal{T}(f, \partial N)$ , and so the relation  $g \underset{K}{\sim} h$  always implies  $g \underset{\partial N}{\sim} h$ .
- (4) Suppose that either
  - $g$  and  $h$  are isotopic as diffeomorphisms of  $M$  or
  - $N$  can be embedded into  $\mathbb{R}^2$ .
 Then  $\mathcal{T}(f, K) = \mathcal{T}(f, \partial N)$  and so the relations  $g \underset{\partial N}{\sim} h$  and  $g \underset{K}{\sim} h$  are equivalent.

*Proof.* Statement (1) is a consequence of [7, Claim 7.1.1], (2) follows from [7, Theorem 6.2 & Lemma 6.4] for Morse maps and from [11, Theorem 5] for all  $f \in \mathcal{F}(M, P)$ , (3) follows from (2), and (4) from [7, Theorem 7.1].  $\square$

Now let  $B$  be a connected component of  $\partial N$ , and  $\widehat{B}$  be a regular component of some level-set of  $f$  such that  $B$  and  $\widehat{B}$  bound a cylinder  $C$  containing no critical points of  $f$ , see Figure 2.2. Denote also  $\widehat{N} = N \cup C$ .

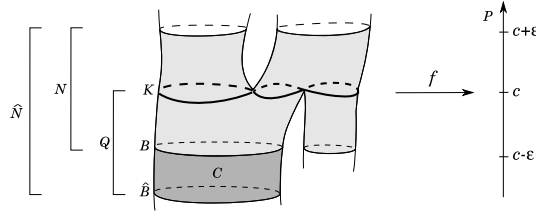


FIGURE 2.2. Extended neighbourhood of  $K$

**Lemma 2.2.** see [7, Lemma 4.14]. Suppose  $K$  is not a local extreme of  $f$ . If  $g, h \in \mathcal{S}(f, \widehat{B})$  are such that

- $g = h$  on some neighbourhood of  $\widehat{N}$  and
- $g$  and  $h$  are isotopic in  $\mathcal{S}(f)$ ,

then they are isotopic in  $\mathcal{S}(f)$  relatively to some neighbourhood of  $\widehat{N}$ .

*Proof.* It suffices to prove this lemma for the case when  $h = \text{id}_M$ , and so we are in the situation when  $g$  belongs to  $\mathcal{S}_{\text{id}}(f)$  and is also fixed on  $N$ .

Suppose  $M$  is orientable. In this case one can define a smooth flow  $\mathbf{F} : M \times \mathbb{R} \rightarrow M$  such that  $g \in \mathcal{S}_{\text{id}}(f)$  if and only if there exists a  $C^\infty$ -function  $\alpha_g : M \rightarrow \mathbb{R}$  satisfying the identity:  $g(x) = \mathbf{F}(x, \alpha_g(x))$  for all  $x \in M$ , see [11, Theorem 3]. Moreover, this function is unique on any connected  $f$ -adopted subsurface containing at least one critical point being not a non-degenerate local extreme of  $f$ . By assumption  $\widehat{N}$  contains such points and  $g$  is fixed on  $\widehat{N}$ . Hence  $\alpha_g = 0$  on  $\widehat{N}$ . Then the isotopy between  $g$  and  $\text{id}_M$  in  $\mathcal{S}_{\text{id}}(f)$  can be given by the formula  $g_t(x) = \mathbf{F}(x, t\alpha_g(x))$ ,  $t \in [0, 1]$ , see [7, Lemma 4.14].

If  $M$  is non-orientable, the proof follows by the arguments similar to the proof of [11, Theorem 3] for non-orientable case.  $\square$

**Lemma 2.3.** *Suppose  $K$  is not a local extreme of  $f$ . Then there exists an epimorphism  $\eta : \mathcal{S}(f, \widehat{B}) \rightarrow \mathbb{Z}$  having the following properties.*

- (1)  $\mathcal{T}(f, K) = \eta^{-1}(m\mathbb{Z})$  for some  $m \geq 1$ . In particular,  $\mathcal{S}(f, \widehat{B})/\mathcal{T}(f, K) \cong \mathbb{Z}_m$ .
- (2) Let  $W$  be any open neighbourhood of  $\widehat{N}$  and  $g, h \in \mathcal{S}(f, \widehat{B})$ . Then  $\eta(g) = \eta(h)$  if and only if there exists an isotopy of  $g$  in  $\mathcal{S}(f, \widehat{B})$  supported in  $W$  to some  $g'$  such that  $g' = h$  on  $\widehat{N}$ .

*Proof.* Let  $V$  be the connected component of  $\widehat{N} \setminus K$  containing  $\widehat{B}$  and  $Q = \overline{V} \setminus \Sigma_f$ . It is easy to see that  $Q$  is diffeomorphic to  $S^1 \times [0, 1] \setminus F$ , where  $F$  is a finite subset of  $S^1 \times 1$ , see Figures 2.2 and 2.3. Let  $p : \widetilde{Q} \rightarrow Q$  be the universal covering map for  $Q$ . Evidently,

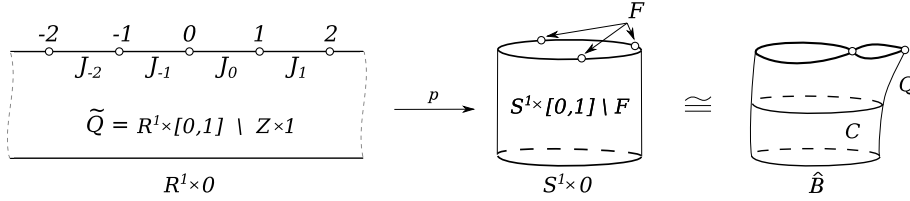


FIGURE 2.3.

$\widetilde{Q}$  is diffeomorphic with  $\mathbb{R} \times [0, 1] \setminus \mathbb{Z} \times 1$ , see Figure 2.3. Let  $b > 0$  be the number of points in  $F$ . It also equals to the number of connected components of  $S^1 \times 1 \setminus F$ . Denote

$$J_i = (i, i + 1) \times 1, \quad i \in \mathbb{Z}.$$

Then

$$p(J_i) = p(J_{i+b}), \quad i \in \mathbb{Z}. \quad (2.1)$$

Now let  $h \in \mathcal{S}(f, \widehat{B})$ . Since  $h$  is fixed on  $\widehat{B}$ , it follows that  $h|_Q$  lifts to a unique diffeomorphism  $\widehat{h}$  of  $\widetilde{Q}$  fixed on  $\mathbb{R}^1 \times 0$ . Then  $\widehat{h}$  “shifts” open intervals  $\{J_i\}_{i \in \mathbb{Z}}$  preserving their linear order. In other words, there exists a unique  $k \in \mathbb{Z}$  such that  $\widehat{h}(J_i) = J_{i+k}$  for all  $i \in \mathbb{Z}$ . Define a map  $\eta' : \mathcal{S}(f, \widehat{B}) \rightarrow \mathbb{Z}$  by

$$\eta'(h) = k.$$

It is easy to check that  $\eta'$  is in fact a homomorphism.

It follows from Eq. (2.1) that  $h \in \mathcal{T}(f, K)$  if and only if  $\eta'(h)$  is divided by  $b$ . In other words

$$\mathcal{T}(f, K) = (\eta')^{-1}(b\mathbb{Z}). \quad (2.2)$$

Let us show that  $\eta'$  is a non-trivial homomorphism. Recall that there exists a Dehn twist  $\tau \in \mathcal{S}(f, \widehat{B})$  supported in  $C$ , see [7, § 6]. Then it is easy to see that  $\eta'(\tau) = b$  or  $-b$ . In particular,  $\tau \in \mathcal{T}(f, K)$ .

Hence the image of  $\eta'$  is also a non-zero subgroup of  $\mathbb{Z}$ , so  $\eta'(\mathcal{S}(f, \widehat{B})) = n\mathbb{Z}$  for some  $n \geq 1$ . In particular, due to Eq. (2.2)  $n$  must divide  $b$ . Therefore the map  $\eta : \mathcal{S}(f, \widehat{B}) \rightarrow \mathbb{Z}$  defined by

$$\eta(h) = \eta'(h)/n$$



is an epimorphism.

Property (1) for  $\eta$  now follows from Eq. (2.2) with  $m = b/n$ . It remains to check (2).

Let  $g, h \in \mathcal{S}(f, \widehat{B})$  and  $\widehat{g}, \widehat{h} : \widetilde{Q} \rightarrow \widetilde{Q}$  be unique liftings of  $g|_Q$  and  $h|_Q$  respectively fixed on  $\mathbb{R}^1 \times 0$ .

Suppose there exists an isotopy  $\{g_t\}_{t \in [0,1]}$  in  $\mathcal{S}(f, \widehat{B})$  such that  $g_0 = g$  and  $g_1 = h$  on  $\widehat{N}$ . We claim that  $\eta(g) = \eta(h)$ .

Indeed, let  $\widehat{g}_t$  be the lifting of  $g_t|_Q$  fixed on  $\mathbb{R}^1 \times 0$ . Then  $\{\widehat{g}_t\}_{t \in [0,1]}$  is an isotopy between  $\widehat{g} = \widehat{g}_0$  and  $\widehat{h} = \widehat{g}_1$ . Hence all  $\widehat{g}_t$  shift boundary components  $\{(i, i+1) \times 1\}_{i \in \mathbb{Z}}$  of  $\widetilde{Q}$  in the same way, and so

$$\eta(g) = \eta(g_0) = \eta(g_t) = \eta(g_1) = \eta(h).$$

Conversely, suppose  $\eta(g) = \eta(h)$ . Then  $g$  and  $h$  interchange edges of  $K$  in the same way, and so by (1) of Lemma 2.1  $g \sim_K h$ . Moreover, by (2) of that lemma  $g$  is isotopic to a diffeomorphism  $g'$  such that  $g' = h$  on  $N$ . Hence  $g' \circ h^{-1}|_{\widehat{N}}$  is supported in a cylinder  $C$ , see Figures 2.2 and 2.3, and so it is isotopic relatively to  $\partial C$  to some degree  $a$  of the Dehn twist  $\tau$  mentioned above. Therefore  $\eta(g' \circ h^{-1}) = \eta(\tau^a) = ak/n$ . However

$$\eta(g' \circ h^{-1}) = \eta(g') - \eta(h) = \eta(g) - \eta(h) = 0,$$

whence  $a = 0$ . This means that  $g' \circ h^{-1}|_C$  is isotopic to  $\tau^0|_C = \text{id}_C$  relatively to  $\partial C$ . Hence by [7, Lemma 4.12(3)] that isotopy can be made  $f$ -preserving. Thus  $g'$  (and therefore  $g$ ) is isotopic in  $\mathcal{S}(f, \widehat{B})$  to some  $g''$  such that  $g'' = h$  on  $\widehat{N}$ . Lemma is completed.  $\square$

### 3. FUNCTIONS ON 2-DISKS AND CYLINDERS

In this section we assume that  $M$  is either a 2-disk or a cylinder,  $f \in \mathcal{F}(M, P)$ , and  $\widehat{B}$  is a connected component of  $\partial M$ . Our aim is to establish the following key result which will be proved in §3.7.

**Proposition 3.1.** *The group  $\pi_0 \mathcal{S}'(f, \partial M)$  belongs to class  $\mathcal{P}$ .*

For the proof we need some preliminary statements.

**Lemma 3.2.** (1)  $\mathcal{S}'(f, \widehat{B}) = \mathcal{S}(f, \widehat{B})$ ,

(2)  $\pi_0 \mathcal{S}'(f, \widehat{B}) = \pi_0 \mathcal{S}'(f, \partial M)$ .

*Proof.* (1) Recall that by definition  $\mathcal{S}'(f, \widehat{B}) := \mathcal{S}(f, \widehat{B}) \cap \mathcal{D}_{\text{id}}(M, \widehat{B})$ . Therefore we should only prove that each  $h \in \mathcal{S}(f, \widehat{B})$  is isotopic to  $\text{id}_M$  relatively to  $\partial M$ . By 5) of Theorem 1.4  $h$  is isotopic in  $\mathcal{S}(f, \widehat{B})$  to a diffeomorphism  $h'$  fixed on some neighbourhood of  $\widehat{B}$ . Since  $M$  is either a 2-disk or a cylinder, it follows from [17, 2] that then  $h'$  is isotopic to  $\text{id}_M$  relatively to  $\widehat{B}$ .

(2) If  $M = D^2$ , then  $\widehat{B} = \partial M$  and the statement is trivial. Suppose  $M = S^1 \times I$ . Then by 1) and 4) of Theorem 1.4 we have the following isomorphisms:

$$\pi_0 \mathcal{S}'(f, \widehat{B}) \cong \pi_1 \mathcal{O}(f, \widehat{B}) \cong \pi_1 \mathcal{O}(f, \partial M) \cong \pi_0 \mathcal{S}'(f, \partial M).$$

Lemma is completed.  $\square$

Thus due to (2) for the proof of Proposition 3.1 it suffices to show that  $\pi_0 \mathcal{S}'(f, \widehat{B}) \in \mathcal{P}$ . Of course, this replacement is non-trivial only for  $M = S^1 \times I$ .

Let  $Z$  be the union of all critical components of all level sets of  $f$ ,  $U$  be a connected component of  $M \setminus Z$  containing  $\widehat{B}$ , and  $K$  be that unique critical component from  $Z$  which intersects  $\overline{U}$ . Roughly speaking,  $K$  is the “closest” to  $\widehat{B}$  critical component of some level set of  $f$ .

Let also  $N$  be an  $f$ -regular neighbourhood of  $K$  that does not contain  $\widehat{B}$  and

$$\widehat{N} = N \cup U, \quad C = \overline{U \setminus N}.$$

Then we are in the notations and under assumptions of Lemma 2.3 for a special case when  $\widehat{B}$  is a boundary component of  $\partial M$ .

By (1) or Lemma 3.2 each  $h \in \mathcal{S}(f, \widehat{B})$  is isotopic to  $\text{id}_M$ , whence by (4) of Lemma 2.1 we get that  $\mathcal{T}(f, K) = \mathcal{T}(f, \partial N)$ . Moreover, by Lemma 2.2 there exists an epimorphism

$$\eta : \mathcal{S}(f, \widehat{B}) \longrightarrow \mathbb{Z}$$

satisfying  $\mathcal{T}(f, K) = \eta^{-1}(m\mathbb{Z})$  for some  $m \geq 1$ , and so

$$\mathcal{S}(f, \widehat{B}) / \mathcal{T}(f, K) = \mathcal{S}(f, \widehat{B}) / \mathcal{T}(f, \partial N) \cong \mathbb{Z}_m.$$

**Lemma 3.3.** *Let  $g \in \mathcal{S}(f, \widehat{B})$  be such that  $g(Y) \cap Y = \emptyset$  for some connected component  $Y$  of  $\overline{M \setminus N}$ . If  $\eta(g) = 1$ , then  $g^i(Y) \cap Y = \emptyset$  for  $i = 1, \dots, m-1$ , and  $g^m(Y) = Y$ .*

*Proof.* Notice that under assumption of lemma  $g \notin \mathcal{T}(f, \partial N)$ , whence  $m > 1$ . Moreover,  $\eta(g^m) = m \in m\mathbb{Z}$ , and so  $g^m \in \mathcal{T}(f, \partial N)$ . Therefore  $g^m(Y) = Y$ .

It remains to consider the case  $i \in \{1, \dots, m-1\}$ . Let  $\widehat{M}$  be a closed surface obtained by gluing every connected component of  $\partial M$  with a 2-disk. Since  $M$  is either a 2-disk or a cylinder, we obtain that  $\widehat{M}$  is a 2-sphere. Then  $\widehat{M} \setminus K$  is a union of open 2-disks, and so we have a cellular subdivision of  $\widehat{M}$  by 0- and 1-cells of  $K$  and 2-cells being connected components of  $\widehat{M} \setminus K$ .

Let  $h \in \mathcal{S}(f, \widehat{N})$ . Since  $h$  leaves invariant boundary components of  $\partial M$ , it extends to a certain homeomorphism  $\widehat{h}$  of all of  $\widehat{M}$  which preserves orientation of  $\widehat{M}$ , and therefore is also homotopic to  $\text{id}_{\widehat{M}}$ . Then by [10, Proposition 5.4] either

- (a) all cells are positively invariant for  $\widehat{h}$ , or
- (b) the number of positively invariant cells of  $\widehat{h}$  is equal to the Euler characteristic of  $\widehat{M}$ , i.e. to 2.

In particular this holds for  $h = g^i$ ,  $i = 1, \dots, m-1$ .

Now let  $\delta_0, \delta_1$  be positively invariant cells of  $\widehat{g}$ . Then they are also positively invariant for  $\widehat{g}^i$ ,  $i = 1, \dots, m-1$ . Notice also that these cells do not intersect  $Y$ , since  $g(Y) \cap Y = \emptyset$ .

Therefore if we assume that  $g^i(Y) = Y$  for some  $i = 1, \dots, m-1$ , then  $\widehat{g}^i$  would have at least 3 positively invariant cells of  $\widehat{M}$ , and by (a) all other cells of  $\widehat{M}$  must also be  $\widehat{g}^i$ -invariant. But this would mean that  $g^i \in \mathcal{T}(f, \partial N)$  which is possible only if  $i$  is a multiple of  $m$ . We get a contradiction with the assumption  $i \in \{1, \dots, m-1\}$ . Hence  $g^i(Y) \cap Y = \emptyset$  for  $1 \leq i \leq m-1$ .  $\square$

Fix any  $g \in \mathcal{S}(f, \widehat{B})$  with  $\eta(g) = 1$  and let

$$C = X_0, X_1, \dots, X_a \quad (3.1)$$

be all  $g$ -invariant connected components of  $\overline{M \setminus N}$ ,

$$X = X_1 \cup \dots \cup X_a$$

be the union of all these components except for  $C$ , and

$$S_i = X_i \cap \partial N, \quad S = X \cap \partial N,$$

see Figure 3.1. By (1) of Lemma 2.3 these notation does not depend on a particular choice of such  $g$ .

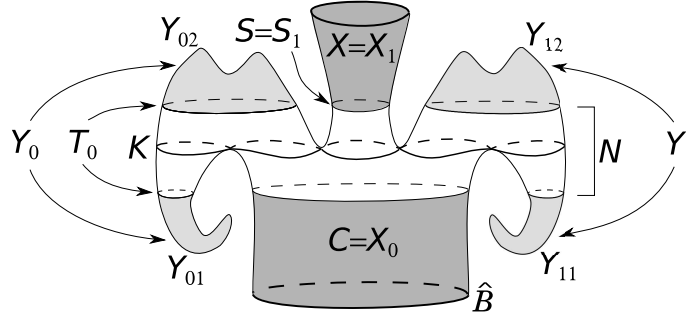


FIGURE 3.1.

**Lemma 3.4.** *There exists  $g \in \mathcal{S}(f, \widehat{B})$  fixed near  $X$  and satisfying  $\eta(g) = 1$ .*

*Proof.* Let  $h \in \mathcal{S}(f, \widehat{B})$  be any element with  $\eta(h) = 1$ . Then  $h$  leaves invariant every connected component of  $S$ , and preserves their orientation. Therefore  $h$  is isotopic in  $\mathcal{S}(f, \widehat{B})$  to a diffeomorphism  $h'$  fixed on some neighbourhood of  $S$ . Now change  $h'$  on  $X$  by the identity and denote the obtained diffeomorphism by  $g$ . Then  $g \in \mathcal{S}(f, \widehat{B})$  and  $\eta(g) = 1$ .  $\square$

Let  $g \in \mathcal{S}(f, \widehat{B})$  be such that  $\eta(g) = 1$ . It follows from Lemma 3.3 that connected components of  $\overline{M \setminus N}$  that are *not*  $g$ -invariant can be enumerated as follows:

$$\begin{array}{cccc} Y_{0,1} & Y_{0,2} & \cdots & Y_{0,b} \\ Y_{1,1} & Y_{1,2} & \cdots & Y_{1,b} \\ \cdots & \cdots & \cdots & \cdots \\ Y_{m-1,1} & Y_{m-1,2} & \cdots & Y_{m-1,b} \end{array} \quad (3.2)$$

so that

$$g(Y_{j,q}) = g(Y_{j+1 \bmod m, q})$$

for all  $j, q$ . In other words,  $g$  cyclically shifts down the rows of Eq. (3.2), see Figure 3.1. Denote

$$\begin{aligned} Y_j &= Y_{j,1} \cup Y_{j,2} \cup \cdots \cup Y_{j,b}, & Y &= \bigcup_{j=0}^{m-1} Y_j, \\ T_{j,q} &= \partial Y_{j,q} \cap N, & T_j &= \partial Y_j \cap N. \end{aligned}$$

Then

$$g^j(Y_0) = Y_j, \quad Y_j \cap Y_{j'} = \emptyset$$

for  $j \neq j' = 0, \dots, m-1$ . Consider also the restrictions

$$f_X = f|_X : X \rightarrow P, \quad f_{Y_j} = f|_{Y_j} : Y_j \rightarrow P.$$

**Lemma 3.5.** *In the notation above there exists an isomorphism*

$$\psi : \pi_0 \mathcal{S}(f, \widehat{B}) \longrightarrow \pi_0 \mathcal{S}(f_X, S) \times (\pi_0 \mathcal{S}(f_{Y_0}, T_0) \wr_{\mathbb{Z}_m} \mathbb{Z}). \quad (3.3)$$

For  $m = 1$ ,  $\psi$  reduces to an isomorphism

$$\psi : \pi_0 \mathcal{S}(f, \widehat{B}) \longrightarrow \pi_0 \mathcal{S}(f_X, S) \times \mathbb{Z}.$$

*Proof.* Choose  $g \in \mathcal{S}(f, \widehat{B})$  fixed near  $X$  and satisfying  $\eta(g) = 1$ , see Lemma 3.4.

Let  $\gamma \in \pi_0 \mathcal{S}(f, \widehat{B})$ . By (1) of Lemma 2.3 we can take a representative  $h \in \gamma$  such that  $g^{-\eta(h)} \circ h$  is fixed on some neighbourhood of  $\widehat{N}$ . As  $g$  is fixed near  $X$ , we have that

$$g^{-\eta(h)} \circ h(X) = h(X) = X, \quad g^{-\eta(h)} \circ h(Y_j) = Y_j,$$

for all  $j$ , whence

$$h|_X \in \mathcal{S}(f_X, S), \quad g^{-j-\eta(h)} \circ h \circ g^j|_{Y_0} \in \mathcal{S}(f_{Y_0}, T_0)$$

for  $j = 1, \dots, m$ . Therefore we obtain a function

$$\sigma_h : \mathbb{Z}_m \longrightarrow \pi_0 \mathcal{S}(f_{Y_0}, T_0), \quad \sigma(j) = [g^{-j-\eta(h)} \circ h \circ g^j|_{Y_0}]$$

$j = 0, \dots, m-1$ .

Consider the following element belonging to  $\pi_0 \mathcal{S}(f_X, S) \times (\pi_0 \mathcal{S}(f_{Y_0}, T_0) \wr_{\mathbb{Z}_m} \mathbb{Z})$ :

$$\psi(\gamma) = ([h|_X], \sigma_h, \eta(h)).$$

We claim that the correspondence  $\gamma \longmapsto \psi(\gamma)$  is the desired isomorphism (3.3).

**Step 1.** First we show that  $\psi(\gamma)$  does not depend on a particular choice of a representative  $h \in \gamma$  such that  $g^{-\eta(h)} \circ h$  is fixed on some neighbourhood of  $\widehat{N}$ .

Indeed, let  $h' \in \gamma$  be another element such that  $g^{-\eta(h')} \circ h'$  is fixed near  $\widehat{N}$ . Then  $h' = h = g^{\eta(h)}$  near  $\widehat{N}$  and  $h'$  is isotopic to  $h$  in  $\mathcal{S}_{\text{id}}(f, \widehat{B})$ .

In particular, it follows from (1) of Lemma 2.3 that  $\eta(h) = \eta(h')$ .

Moreover, by Lemma 2.2  $h$  and  $h'$  are isotopic in  $\mathcal{S}_{\text{id}}(f, \widehat{B})$  relatively some neighbourhood of  $\widehat{N}$ . This implies that  $h|_X$  is isotopic to  $h'|_X$  relatively some neighbourhood of  $S$ , and for each  $j = 0, \dots, m-1$  the restriction  $g^{-j-\eta(h)} \circ h \circ g^j|_{Y_0}$  is isotopic to  $g^{-j-\eta(h)} \circ h' \circ g^j|_{Y_0}$  relatively some neighbourhood of  $T_0$ . In other words,

$$[h|_X] = [h'|_X] \in \pi_0 \mathcal{S}'(f_X, S),$$

$$[g^{-j-\eta(h)} \circ h \circ g^j|_{Y_0}] = [g^{-j-\eta(h)} \circ h' \circ g^j|_{Y_0}] \in \pi_0 \mathcal{S}(f_{Y_0}, T_0),$$

$j = 0, \dots, m-1$ . Hence  $\psi(\gamma)$  does not depend on a particular choice of such  $h$ .

**Step 2.**  $\psi$  is a homomorphism. Let  $h_0, h_1 \in \mathcal{S}(f, \widehat{B})$ . We have to show that

$$\psi([h_0 \circ h_1]) = \psi([h_0]) \cdot \psi([h_1]).$$

Put  $k_i = \eta(h_i)$ ,  $i = 0, 1$ . Since  $\eta$  is a homomorphism,  $\eta(h_0 \circ h_1) = k_0 + k_1$ .

By Step 1 we can assume that  $g^{-k_i} \circ h_i$  is fixed on  $\widehat{N}$ ,  $i = 0, 1$ . Define the following four functions

$$\sigma_0, \sigma_1, \sigma, \bar{\sigma} : \mathbb{Z}_m \longrightarrow \pi_0 \mathcal{S}(f_{Y_0}, T_0)$$

by

$$\begin{aligned} \sigma_0(j) &= [g^{-j-k_0} \circ h_i \circ g^j|_{Y_0}], & \sigma_1(j) &= [g^{-j-k_1} \circ h_i \circ g^j|_{Y_0}], \\ \sigma(j) &= [g^{-j-k_0-k_1} \circ h_0 \circ h_1 \circ g^j|_{Y_0}], & \bar{\sigma}(j) &= \sigma_0(j+k_1) \circ \sigma_1(j) \end{aligned}$$

for  $j = 0, \dots, m-1$ . Then

$$\begin{aligned} \psi([h_i]) &= ([h_i|_X], \sigma_i, k_i), \quad i = 0, 1, \\ \psi([h_0 \circ h_1]) &= ([h_0 \circ h_1|_X], \sigma, k_0 + k_1), \end{aligned}$$

and by the definition of multiplication

$$\begin{aligned} \psi([h_0]) \circ \psi([h_1]) &= ([h_0|_X], \sigma_0, k_0) ([h_1|_X], \sigma_1, k_1) \\ &= ([h_0|_X] \circ [h_1|_X], \bar{\sigma}, k_0 + k_1) = ([h_0 \circ h_1|_X], \bar{\sigma}, k_0 + k_1). \end{aligned}$$

It remains to show that  $\bar{\sigma} = \sigma$ . Let  $j = 0, \dots, m-1$ , then

$$\begin{aligned} \sigma(j) &= [g^{-j-k_0-k_1} \circ h_0 \circ h_1 \circ g^j|_{Y_0}] \\ &= [g^{-(j+k_1)-k_0} \circ h_0 \circ g^{j+k_1}|_{Y_0}] \circ [g^{-j-k_1} \circ h_1 \circ g^j|_{Y_0}] \\ &= \sigma_0(j+k_1) \circ \sigma_1(j) = \bar{\sigma}(j). \end{aligned}$$

Thus  $\psi$  is a homomorphism.

**Step 3.**  $\psi$  is a monomorphism. Let  $h \in \mathcal{S}(f, \widehat{B})$  be such that  $g^{-\eta(h)} \circ h$  is fixed near  $\widehat{N}$ , and suppose that  $[h] \in \ker(\psi)$ . This means that

$$\begin{aligned} [h|_X] &= [\text{id}_X] \in \pi_0 \mathcal{S}(f_X, S), \\ [g^{-j} \circ h \circ g^j|_{Y_0}] &= [\text{id}_{Y_0}] \in \pi_0 \mathcal{S}(f_{Y_0}, T_0), \\ \eta(h) &= 0, \end{aligned}$$

for  $j = 0, \dots, m-1$ . In other words,  $h|_X$  is isotopic in  $\mathcal{S}_{\text{id}}(f_X, S)$  to  $\text{id}_X$ , and  $h|_{Y_j}$  is isotopic in  $\mathcal{S}_{\text{id}}(f_{Y_j}, T_j)$  to  $\text{id}_{Y_j}$ . These isotopies give an isotopy between  $h$  and  $\text{id}_M$  in  $\mathcal{S}(f, \widehat{B})$ . Hence  $[h] = [\text{id}_M] \in \mathcal{S}(f, \widehat{B})$ , and so  $\ker(\psi)$  is trivial.

**Step 4.**  $\psi$  is surjective. Let  $\widehat{h} \in \mathcal{S}(f_X, S)$ ,  $\sigma : \mathbb{Z}_m \rightarrow \pi_0 \mathcal{S}(f_{Y_0}, T_0)$ , and  $k \in \mathbb{Z}$ . We have to find  $h \in \mathcal{S}(f, \widehat{B})$  with  $\psi([h]) = ([h], \sigma, k)$ . For each  $j \in \mathbb{Z}_m$  choose  $h_j \in \mathcal{S}(f_{Y_0}, T_0)$  such that  $\sigma(j) = [h_j]$ . Due to 5) of Theorem 1.4 we can assume that  $\widehat{h}$  is fixed near  $S$  and each  $h_j$  is fixed near  $T_0$ . Define  $h$  by the formula:

$$h(x) = \begin{cases} g^k(x), & x \in N, \\ g^k \circ \widehat{h}(x), & x \in X, \\ g^{j+k} \circ h_j \circ g^{-j}(x), & x \in Y_j, \quad j = 0, \dots, m-1. \end{cases}$$

Then it is easy to check that  $h \in \mathcal{S}(f, \widehat{B})$  and  $\psi([h]) = ([\widehat{h}], \sigma, k)$ . Lemma 3.5 is completed.  $\square$

**Lemma 3.6.** 1) *Let  $f \in \mathcal{F}(S^1 \times I, P)$  be a map without critical points. Then*

$$\pi_0 \mathcal{S}'(f, S^1 \times 0) = \pi_0 \mathcal{S}'(f, \partial(S^1 \times I)) = 0.$$

2) *Let  $f \in \mathcal{F}(D^2, P)$  be a map having exactly one critical point, which therefore must be a local extreme.*

(a) *If  $z$  is a **non-degenerate** local extreme of  $f$ , then  $\pi_0 \mathcal{S}'(f, \partial D^2) = 0$ .*

(b) *Suppose  $z$  is a **degenerate** local extreme of  $f$ . Then  $\pi_0 \mathcal{S}'(f, \partial D^2) \cong \mathbb{Z}$ .*

*Proof.* These statements are contained in the previous papers by the author, though they were not explicitly formulated. In fact, statement 1) follows from [7, Lemma 4.12(2,3)], statement 2(a) from [5, Eq (25)] or from results of [9, 6], and statement 2(b) from results of [8]. We leave the details to the reader.  $\square$

**3.7. Proof of Proposition 3.1.** Due to (2) of Lemma 3.2 it suffices to prove that  $\pi_0 \mathcal{S}'(f, \widehat{B}) \in \mathcal{P}$ .

If  $K$  is either empty or consists of a unique point, then by Lemma 3.6  $\pi_0 \mathcal{S}'(f, \widehat{B})$  is either trivial or isomorphic with  $\mathbb{Z}$ . Therefore it belongs to the class  $\mathcal{P}$ .

Suppose now that  $K$  consists of more than one point, and let  $n$  be the total number of critical points of  $f$  in all of  $M$ . We will use induction on  $n$ .

If  $n = 0$ , then we are in the case 1) of Lemma 3.6 which is already considered. Suppose Proposition 3.1 is proved for all  $n < k$  for some  $k \geq 1$ . Let us establish it for  $n = k$ .

Preserving notation of Lemma 3.5 let  $K$  be the “closest” to  $\widehat{B}$  critical component of some level set of  $f$ , see beginning of §3. Since  $X$  is a disjoint union of surfaces  $X_i$ ,  $i = 1, \dots, a$ , as well as  $Y_0$  is a disjoint union of  $Y_{0,q}$ ,  $q = 1, \dots, b$ , it follows that

$$\pi_0 \mathcal{S}(f_X, S) \cong \bigtimes_{i=1}^a \pi_0 \mathcal{S}(f_{X_i}, S_i), \quad \pi_0 \mathcal{S}'(f_{Y_0}, S_0) \cong \bigtimes_{q=1}^b \pi_0 \mathcal{S}(f_{Y_{0,q}}, T_{0,q}),$$

whence from Lemma 3.5 we get an isomorphism

$$\pi_0 \mathcal{S}(f, \widehat{N}) \cong \left( \bigtimes_{i=1}^a \pi_0 \mathcal{S}(f_{X_i}, S_i) \right) \times \left( \left( \bigtimes_{q=1}^b \pi_0 \mathcal{S}(f_{Y_{0,q}}, T_{0,q}) \right) \wr_{\mathbb{Z}_m} \mathbb{Z} \right).$$

As each pair  $(M, \widehat{B})$ ,  $(X_i, S_i)$ , and  $(Y_{j,q}, T_{j,q})$  is diffeomorphic either with  $(D^2, \partial D^2)$  or with  $(S^1 \times I, S^1 \times 0)$ , it follows from (1) of Lemma 3.2 that

$$\mathcal{S}'(f, \widehat{N}) = \mathcal{S}(f, \widehat{N}), \quad \mathcal{S}'(f_{X_i}, S_i) = \mathcal{S}(f_{X_i}, S_i), \quad \mathcal{S}'(f_{Y_{0,q}}, T_{0,q}) = \mathcal{S}(f_{Y_{0,q}}, T_{0,q}),$$

so we also have an isomorphism

$$\pi_0 \mathcal{S}'(f, \widehat{N}) \cong \left( \bigtimes_{i=1}^a \pi_0 \mathcal{S}'(f_{X_i}, S_i) \right) \times \left( \left( \bigtimes_{q=1}^b \pi_0 \mathcal{S}'(f_{Y_{0,q}}, T_{0,q}) \right) \wr_{\mathbb{Z}_m} \mathbb{Z} \right).$$

Notice that each of the restrictions  $f|_{X_i}$  and  $f|_{Y_{0,q}}$  has less critical points than  $n$ , whence by inductive assumption  $\pi_0 \mathcal{S}'(f_{X_i}, S_i)$  and  $\pi_0 \mathcal{S}'(f_{Y_{0,q}}, T_{0,q})$  belong to the class  $\mathcal{P}$ . Hence  $\pi_0 \mathcal{S}'(f, \widehat{N}) \in \mathcal{P}$  as well. Proposition 3.1 is completed.

## 4. PROOF OF THEOREM 1.10

Let  $M$  be a compact orientable surface distinct from  $S^2$  and  $T^2$ . Then  $\mathcal{D}_{\text{id}}(M, \partial M)$  is contractible, and by 1) and 4) of Theorem 1.4 for each  $f \in \mathcal{F}(M, P)$  we have the following isomorphisms

$$\pi_1 \mathcal{O}(f) \cong \pi_1 \mathcal{O}(f, \partial M) \cong \pi_0 \mathcal{S}'(f, \partial M).$$

Therefore it suffices to prove that the class  $\mathcal{P}$  coincides with each of the following classes of groups:

$$\{ \pi_0 \mathcal{S}'(f, \partial M) \mid f \in \text{Morse}(M, P) \}, \quad \{ \pi_0 \mathcal{S}'(f, \partial M) \mid f \in \mathcal{F}(M, P) \}.$$

**Lemma 4.1.** *For each  $f \in \mathcal{F}(M, P)$  the group  $\pi_0 \mathcal{S}'(f, \partial M)$  belongs to  $\mathcal{P}$ .*

*Proof.* By 4) of Theorem 1.4 there exist finitely many disjoint subsurfaces  $X_1, \dots, X_n \subset M$  each  $X_i$  is diffeomorphic either with  $D^2$  or with  $S^1 \times I$ , and such that

$$\pi_0 \mathcal{S}'(f, \partial M) \cong \bigtimes_{i=1}^n \pi_0 \mathcal{S}'(f_{X_i}, \partial X_i).$$

But by Proposition 3.1  $\pi_0 \mathcal{S}'(f_{X_i}, \partial X_i) \in \mathcal{P}$  for all  $i$ , whence  $\pi_0 \mathcal{S}'(f, \partial M) \in \mathcal{P}$  as well.  $\square$

For the converse statement we make a remark concerning the structure of groups from  $\mathcal{P}$ . By definition a group  $G$  belongs to the class  $\mathcal{P}$  if and only if it can be obtained from the unit group  $\{1\}$  by finitely many operations of direct product  $\times$  and wreath product  $\wr \mathbb{Z}$  from the top with  $\mathbb{Z}$ . We will call such a presentation of  $G$  a  $\mathcal{P}$ -presentation.

$\mathbb{Z}_m$  A priori a  $\mathcal{P}$ -presentation of  $G$  is not unique, e.g.  $\mathbb{Z} \cong 1 \wr_{\mathbb{Z}_1} \mathbb{Z} \cong 1 \wr_{\mathbb{Z}_3} \mathbb{Z}$ . Given a  $\mathcal{P}$ -presentation  $\xi_G$  of  $G$  denote by  $\mu(\xi_G)$  the total number of signs  $\times$  and  $\wr_{\mathbb{Z}_m} \mathbb{Z}$  for some  $m \geq 1$ , used in  $\xi_G$ . For example, a group  $G = \mathbb{Z}^2 \times (\mathbb{Z} \wr_{\mathbb{Z}_4} \mathbb{Z})$  has a  $\mathcal{P}$ -presentation

$$\xi_G : G \cong (1 \wr_{\mathbb{Z}_1} \mathbb{Z}) \times (1 \wr_{\mathbb{Z}_1} \mathbb{Z}) \times ((1 \wr_{\mathbb{Z}_1} \mathbb{Z}) \wr_{\mathbb{Z}_4} \mathbb{Z}).$$

with  $\mu(\xi_G) = 6$ .

**Lemma 4.2.** *For each  $G \in \mathcal{P}$  then there exists an  $f \in \text{Morse}(M, P)$  such that*

$$\pi_0 \mathcal{S}'(f, \partial M) \cong G.$$

*Proof. Case  $M = D^2$  or  $S^1 \times I$ .* If  $G = \{1\}$  is a unit group, we take  $f$  to be a Morse map from 1) or 2a) of Lemma 3.6 according to  $M$ . Then  $\pi_0 \mathcal{S}'(f, \partial M) \cong G = \{1\}$ .

Suppose that we proved our lemma for all groups  $A \in \mathcal{P}$  having a  $\mathcal{P}$ -presentation  $\xi_A$  with  $\mu(\xi_A) < n$  and let  $G \in \mathcal{P}$  be a group having a  $\mathcal{P}$ -presentation  $\xi_G$  with  $\mu(\xi_G) = n$ . It follows from the definition of class  $\mathcal{P}$  that then either

- (i) there exist  $A, B \in \mathcal{P}$  and  $m \geq 2$ , such that  $G \cong A \times (B \wr_{\mathbb{Z}_m} \mathbb{Z})$ , where  $A$  and  $B$  have  $\mathcal{P}$ -presentations  $\xi_A$  and  $\xi_B$  with  $\mu(\xi_A), \mu(\xi_B) < \mu(\xi_G)$ , or
- (ii) there exist  $A \in \mathcal{P}$  such that  $G \cong A \times \mathbb{Z}$ , where  $A$  has a  $\mathcal{P}$ -presentation  $\xi_A$  with  $\mu(\xi_A) < \mu(\xi_G)$ .

First assume that  $M = D^2$ .

(i) Suppose  $G \cong A \times (B \wr \mathbb{Z})$ . Define a Morse function  $\varphi : M \rightarrow P$ , as it is shown in Figure 4.1(a) for  $m = 3$ . So  $\varphi$  has one local minimum  $x$  and  $m$  local maximums  $y_0, \dots, y_{m-1}$  satisfying  $\varphi(y_0) = \dots = \varphi(y_{m-1})$  and there exists a diffeomorphism  $g \in \mathcal{S}(f, \partial M)$  that cyclically interchange these points, i.e.  $g(y_j) = y_{j+1 \bmod m}$ . Let  $X$  be a  $\varphi$ -regular disk neighbourhood of  $x$ ,  $Y_0$  be a  $\varphi$ -regular disk neighbourhood of  $y_0$ , and  $Y_j = g^j(Y_0)$ ,  $j = 1, \dots, m-1$ . As  $\mu(\xi_A), \mu(\xi_B) < \mu(\xi_G)$ , we have by induction that there

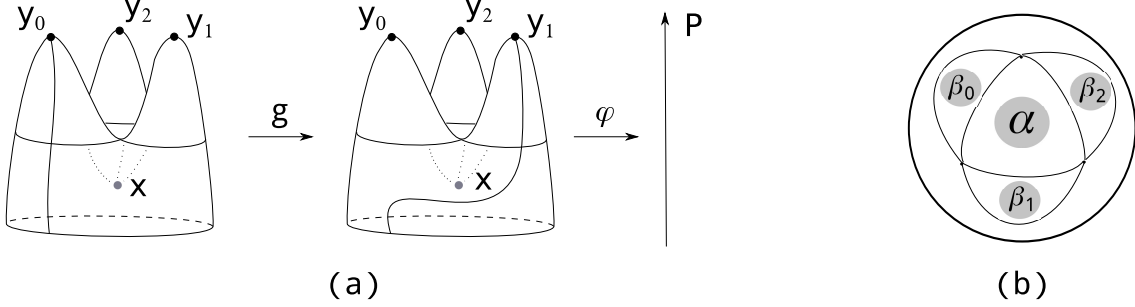


FIGURE 4.1.  $M = D^2$ . Case (i)

exist  $\alpha \in \text{Morse}(X, P)$  and  $\beta \in \text{Morse}(Y_0, P)$  such that

$$A \cong \pi_0 \mathcal{S}'(\alpha, \partial X), \quad B \cong \pi_0 \mathcal{S}'(\beta, \partial Y_0).$$

Not loosing generality, one can assume that  $\alpha = \varphi$  in a neighbourhood of  $\partial X$  and  $\beta = \varphi$  in a neighbourhood of  $\partial Y$ . Replace  $\varphi$  with  $\alpha$  on  $X$ , with  $\beta_j = \beta \circ g^{-j}$  on  $Y_j$ ,  $j = 0, \dots, m-1$ , and denote the obtained new map by  $f$ , see Figure 4.1(b). Then  $f \in \text{Morse}(M, P)$  and it follows from Proposition 3.1 that

$$\pi_0 \mathcal{S}'(f, \partial M) \cong A \times (B \wr \mathbb{Z}) \cong G.$$

(ii) Suppose now  $G \cong A \times \mathbb{Z}$ . Define a Morse function  $\varphi : M \rightarrow P$  having two local maximums  $x$  and  $y$  such that  $\varphi(x) \neq \varphi(y)$ , see Figure 4.2(a). Let  $Y$  be a  $\varphi$ -regular disk

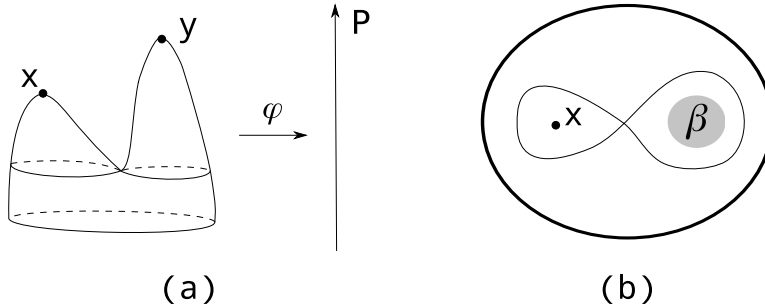


FIGURE 4.2.  $M = D^2$ . Case (ii)

neighbourhood of  $y$  such that  $\varphi(x) \notin \varphi(Y)$ . Since  $\mu(\xi_A) < \mu(\xi_G)$ , it follows by induction that there exist  $\beta \in \text{Morse}(Y, P)$  such that  $A \cong \pi_0 \mathcal{S}'(f, \partial Y)$  and  $\alpha = \varphi$  near  $\partial Y$ . Now



replace  $\varphi$  with  $\beta$  on  $Y$  and denote the obtained map by  $f$ . Then  $f \in \text{Morse}(M, P)$  and it follows from Proposition 3.1 and Lemma 3.6 2(a) that

$$\pi_0 \mathcal{S}'(f, \partial M) \cong A \times \mathbb{Z} \cong G.$$

For  $M = S^1 \times I$ , the proof of the cases (i) and (ii) is similar to the case of  $D^2$ , and is illustrated in Figure 4.3. We leave the details for the reader.

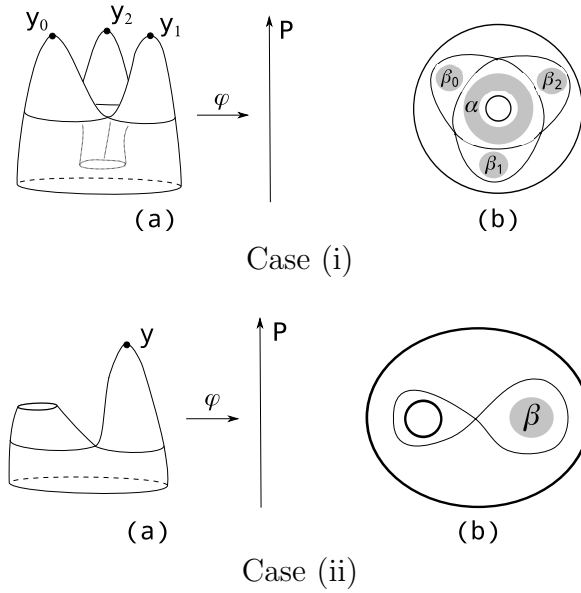


FIGURE 4.3.  $M = S^1 \times I$

Now let  $M$  be an arbitrary compact orientable surface distinct from  $S^2$ ,  $T^2$ ,  $D^2$ , and  $S^1 \times I$ . Choose a Morse function  $\varphi : M \rightarrow P$  such that

- all critical points of  $\varphi$  of index 1 belongs to the same critical level-set of  $\varphi$ ;
- the values of  $\varphi$  at distinct boundary components and distinct local extremes of  $\varphi$  are distinct,

see Figure 4.4. Fix some local extreme  $y$  of  $\varphi$  and let  $Y$  be a  $\varphi$ -regular disk neighbourhood

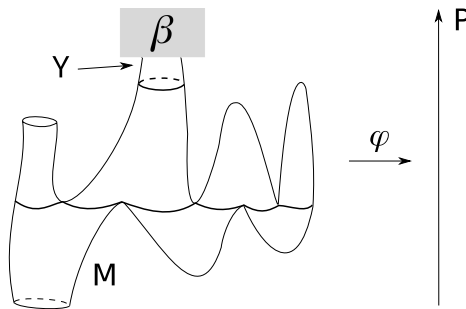


FIGURE 4.4. General case

of  $y$ .

Let  $G \in \mathcal{P}$ . Since the theorem is already proved for a disk  $D^2 \simeq Y$ , there exists  $\beta \in \mathcal{F}(Y, P)$  with  $\pi_0 \mathcal{S}'(\beta, \partial Y) \cong G$  and  $\beta = \varphi$  in some neighbourhood of  $\partial Y$ . Replace  $\varphi$  with  $\beta$  on  $Y$  and denote the obtained map by  $f$ . Then  $f \in \text{Morse}(M, P)$  and it follows from Proposition 3.1 that  $\pi_0 \mathcal{S}'(\varphi, \partial M) \cong \pi_0 \mathcal{S}'(f, \partial Y) \cong G$ . Lemma 4.2 and Theorem 1.10 completed.  $\square$

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